

Complete Conjugacy Invariants of Nonlinearizable Holomorphic Dynamics

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Abstract. *Perez-Marco proved the existence of non-trivial totally invariant connected compacts called hedgehogs near the fixed point of a nonlinearizable germ of holomorphic diffeomorphism. We show that if two nonlinearisable holomorphic germs with a common indifferent fixed point have a common hedgehog then they must commute. This allows us to establish a correspondence between hedgehogs and nonlinearizable maximal abelian subgroups of $\text{Diff}(\mathbf{C}, \mathbf{0})$. We also show that two nonlinearizable germs are conjugate if and only if their rotation numbers are equal and a hedgehog of one can be mapped conformally onto a hedgehog of the other. Thus the conjugacy class of a nonlinearizable germ is completely determined by its rotation number and the conformal class of its hedgehogs.*

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1. Introduction.

We consider the dynamics of a holomorphic germ $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{R} - \mathbf{Q}$ near the indifferent irrational fixed point 0. The germ f is said to be *linearizable* if there is a holomorphic change of variables $z = h(w) = w + O(w^2)$ such that

$$h^{-1} \circ f \circ h = R_\alpha,$$

where $R_\alpha(w) = e^{2\pi i\alpha}w$ is the rigid rotation. The maximal linearization domain of f is called the Siegel disk of f .

The problem of *linearization*, or determining when f is linearizable, is intimately linked to the arithmetic of the rotation number α , and has a long and interesting history. The work of H. Cremer ([Cr1], [Cr2]) in the 1920's showed the existence of nonlinearizable germs for rotation numbers very well approximable by rationals, while that of C.L. Siegel ([Si]) in 1942 and A.D. Brjuno in the 1960's showed linearization was always possible for germs with rotation numbers poorly approximated by rationals. The matter was settled definitively by J.C. Yoccoz ([Yo]) in 1987, when he showed that Brjuno's arithmetic condition was the optimal one for linearizability. The reader is referred to R. Perez-Marco's Bourbaki Seminar [PM4] for a complete account of the story.

When f is linearisable, the closures of the linearization domains $h(\{|w| < r\})$ for r small are completely invariant connected compacts for f . It is not obvious however whether nonlinearizable germs have completely invariant non-trivial connected compacts,

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but Perez-Marco showed ([PM1]) that in fact for any germ f there are always completely invariant, non-trivial connected compacts near the fixed point. These invariant compacts K are called *Siegel compacta*. If K is not contained in the closure of a linearisation domain it is called a *hedgehog*. The *hedgehog* is called linearizable or non-linearizable depending on whether it contains a linearization domain or not.

Perez-Marco has studied the topology (in [PM2]) and the dynamics (in [PM3]) of hedgehogs. The results in [PM2] show that the topology of hedgehogs is complex. Non-linearizable hedgehogs have empty interior. They are not locally connected at any point except possibly at the fixed point. They always contain points inaccessible from their complement.

The results in [PM3] show that the dynamics on hedgehogs has many features in common with linearizable dynamics. For example, there are no periodic points on the hedgehog, and every point is recurrent (as is the case in the linearizable situation). Any hedgehog contains a continuous nested one-parameter family of sub-hedgehogs (recalling the filtration of a Siegel disk by invariant sub-disks). If two germs commute then they preserve the same hedgehogs (commuting linearizable germs have common linearization domains).

These results suggest that hedgehogs should be thought of as "degenerate linearization domains" in some sense, though we are unable at present to give this heuristic notion a precise mathematical formulation. Nevertheless it is one of the main motivations behind Perez-Marco's results and the results of this article. We prove the converse of the last result mentioned above. If two germs preserve a common hedgehog then they must commute (recall that linearizable germs with a common linearization domain must commute). So the group of germs preserving the hedgehog is commutative, and equal to the centralizer in $\text{Diff}(\mathbf{C}, 0)$ of any element of the group. As Perez-Marco shows in [PM3], there is a unique continuous nested one-parameter family of hedgehogs associated to a nonlinearisable germ, indeed to its centralizer since commuting germs have the same hedgehogs. On the other hand since germs preserving the same hedgehogs commute, to every family of hedgehogs we can associate an abelian subgroup of germs. We show that this gives a one-to-one correspondence between hedgehogs and the nonlinearizable maximal abelian subgroups of $\text{Diff}(\mathbf{C}, 0)$ (nonlinearizable meaning not conjugate to the subgroup $(R_\lambda(z) = \lambda z)_{\lambda \in \mathbf{C}^*}$).

The correspondence is natural with respect to change of variables: two such subgroups are conjugate by an element ϕ of $\text{Diff}(\mathbf{C}, 0)$ if and only if ϕ maps one family of hedgehogs to the other. Two nonlinearisable germs are conjugate if and only if they have the same rotation number and a hedgehog of one can be 'conformally mapped' to the other (ie there is a conformal mapping between neighbourhoods of the hedgehogs taking one hedgehog to the other). Thus the geometry of the hedgehog (in some sense its conformal geometry) completely determines the nonlinearizable dynamics. It would be interesting (and morally satisfying) to find an intrinsic notion of conformal structure for singular spaces like hedgehogs characterizing their conformal equivalence. It is interesting to compare this with the description of conjugacy classes of germs tangent to the identity given by Martinet-Ramis in [Ma-Ra]. They describe a complete set of invariants as a formal invariant λ (a complex number) together with a singular quotient object, the "chapelet de spheres", and define an

appropriate notion of conformal equivalence of these objects, so that the conjugacy class of a germ is determined by the conformal class of the associated chaquet.

2. Preliminaries.

Here we collect the basic definitions and results about hedgehogs due to Perez-Marco which we will require. We let $f(z) = e^{2\pi i\alpha}z + O(z^2)$, $\alpha \in \mathbf{R}$, be a germ of holomorphic diffeomorphism with an indifferent fixed point at 0.

Definition 2.1 (Admissible Domain). An *admissible domain* for f is a Jordan domain with C^1 -boundary U containing 0 such that f and f^{-1} extend univalently to a neighbourhood of \overline{U} .

Definition 2.2 (Siegel Compacta, Hedgehogs). A *Siegel compact* of f is a full, connected, compact set K strictly containing 0 such that f and f^{-1} extend univalently to a neighbourhood of K and leave K invariant, $f(K) = f^{-1}(K) = K$. When the rotation number α is irrational, a Siegel compact which is not contained in the closure of a linearization domain of f is called a *hedgehog*. A hedgehog is called linearizable if it contains a linearization domain and nonlinearizable otherwise.

Theorem 2.3 (Existence and Uniqueness of Siegel compacta) ([PM1]). *For any admissible domain U there is a Siegel compact K contained in \overline{U} which extends upto the boundary of U , i.e. $K \cap \partial U \neq \emptyset$. When the rotation number α is irrational then there is a unique such Siegel compact $K = K(U)$ which is in fact equal to the connected component containing 0 of the set of non-escaping points $\{z \in \overline{U} : f^n(z) \in \overline{U} \ \forall n \in \mathbf{Z}\}$ of U . We call $K(U)$ the Siegel compact associated to U .*

We now restrict ourselves to the case of germs with irrational rotation number (so any Siegel compact is either a linearization domain, or a linearizable hedgehog, or a nonlinearizable hedgehog). If two admissible domains are nested, $U \subset V$, then the non-escaping points of U are of course non-escaping points of V , and it follows from the above Theorem that the associated Siegel compacta are nested, $K(U) \subset K(V)$. In fact any Siegel compact is filtered by a nested family of sub-Siegel compacta:

Theorem 2.4 ([PM3]). *Let K be a Siegel compact. Given an admissible neighbourhood U such that $K = K(U)$ and a continuous monotone increasing one-parameter family of admissible neighbourhoods $(U_t)_{0 < t \leq 1}$, $\cap_t U_t = \{0\}$, $\cup_t U_t = U_1 = U$, the associated family of Siegel compacta $(K_t = K(U_t))_{0 < t \leq 1}$ is a continuous (for the Hausdorff topology on compact sets), monotone increasing family of sub-Siegel compacta of K , such that $K_t \rightarrow \{0\}$ as $t \rightarrow 0$ and $K_t \rightarrow K_1 = K$ as $t \rightarrow 1$. Moreover any Siegel compact contained in \overline{U} (in particular any sub-Siegel compact of K) belongs to the family $(K_t)_{0 < t \leq 1}$.*

Thus the sub-Siegel compacta of a given Siegel compact form a continuous, monotone, one-parameter family (which is a trivial fact for linearization domains, but quite a remarkable one for hedgehogs, given their complex topological structures). The parametrization of the family is not unique; different choices of the admissible domain U and the filtration

$(U_t)_{0 < t \leq 1}$ lead to different parametrizations. In the case of a linearizable germ and a linearization domain, the sub-linearization domains admit a natural parametrization given by the conformal radius, which is conformally invariant. An interesting problem is to find such a conformally invariant parametrization for sub-hedgehogs of a hedgehog; is there an appropriate notion of conformal radius for hedgehogs?

Any linearizable hedgehog of a germ f must contain the Siegel disk of f ; however, it can have no other interior points:

Theorem 2.5 ([PM3]) *The interior of a linearizable hedgehog is equal to the Siegel disk. The interior of a nonlinearizable hedgehog is empty.*

The family of all Siegel compacta of f ,

$$\mathcal{K}(f) = \{ K : K \text{ is a Siegel compact of } f \},$$

is a linearly ordered family with respect to inclusion: given Siegel compacta K_1, K_2 , they are both sub-Siegel compacta of their union $K_1 \cup K_2$ (which is full, since by the maximum principle and Theorem 2.5 the complement cannot have any bounded components) and hence by Theorem 2.4 one must be contained in the other. For a nonlinearizable germ f the union $\cup_{K \in \mathcal{K}(f)} K$ of all its hedgehogs is, morally speaking, the analogue of the Siegel disk; note that the closure of this union is *not* a hedgehog, since every hedgehog is strictly contained in some admissible domain and hence in a larger Siegel compact.

While the family of all Siegel compacta $\mathcal{K}(f)$ of a germ f is a natural object to consider, it is not an invariant of the local dynamics; for any germ there are of course local changes of variables which do not extend univalently to neighbourhoods of all the Siegel compacta of the germ. However, since any change of variables is univalent on a neighbourhood of every sufficiently small Siegel compact, the situation is easily remedied by considering the germ of the family of Siegel compacta.

Definition 2.6 (Germ of Siegel compact, Germ of Hedgehog). Consider all families \mathcal{K} of compacta in the plane such that $0 \in K$ for all $K \in \mathcal{K}$, and such that for any $\epsilon > 0$ there is a $K \in \mathcal{K}$ with $\text{diam}(K) < \epsilon$. We define the following equivalence relation on such families:

$\mathcal{K}_1 \sim \mathcal{K}_2$ if there is an $\epsilon = \epsilon(\mathcal{K}_1, \mathcal{K}_2) > 0$ such that for any compact K with $\text{diam}(K) < \epsilon$, $K \in \mathcal{K}_1 \Leftrightarrow K \in \mathcal{K}_2$.

We call an equivalence class $[\mathcal{K}]$ a *germ of compact*.

Given a germ $f(z) = e^{2\pi i \alpha} z + O(z^2)$, $\alpha \in \mathbf{R} - \mathbf{Q}$, the *germ of Siegel compact* of f is defined to be the germ of compact $[\mathcal{K}(f)]$ (where $\mathcal{K}(f)$ is the family of all Siegel compacta of f). If f is nonlinearizable, we call $[\mathcal{K}(f)]$ a *germ of hedgehog*.

There is a natural action of germs of homeomorphisms of \mathbf{C} fixing 0 on germs of compacta,

$$(\phi, [\mathcal{K}] \mapsto \phi([\mathcal{K}]))$$

where $\phi([\mathcal{K}])$ is defined as follows: it is possible to pick a representative $\mathcal{K}' \sim \mathcal{K}$ such that ϕ is defined in a neighbourhood of all $K \in \mathcal{K}'$. Set $\phi([\mathcal{K}]) = [\{ \phi(K) : K \in \mathcal{K}' \}]$. It is easy to see this gives a well-defined action.

The action restricts to an action of $\text{Diff}(\mathbf{C}, 0)$ on germs of Siegel compacta:

Proposition 2.7. *Any germ of diffeomorphism $\phi \in \text{Diff}(\mathbf{C}, 0)$ takes germs of Siegel compacta to germs of Siegel compacta; indeed $\phi([\mathcal{K}(f)]) = [\mathcal{K}(\phi \circ f \circ \phi^{-1})]$. Thus germs of Siegel compacta are holomorphic conjugacy invariants.*

Proof: Given ϕ in $\text{Diff}(\mathbf{C}, 0)$ and a germ of Siegel compact $[\mathcal{K}(f)]$, pick a representative $\mathcal{K} \subseteq \mathcal{K}(\{f\})$ of $[\mathcal{K}(f)]$ such that ϕ is univalent in a neighbourhood of all $K \in \mathcal{K}$. Let $\phi(\mathcal{K})$ be the family of compacta $\{\phi(K) : K \in \mathcal{K}\}$. Since ϕ takes invariant sets of f to invariant sets of the conjugate $\phi \circ f \circ \phi^{-1}$, $\phi(\mathcal{K})$ is a family of Siegel compacta of $\phi \circ f \circ \phi^{-1}$, which moreover contains all sufficiently small Siegel compacta of $\phi \circ f \circ \phi^{-1}$, so $\phi(\mathcal{K}) \sim \mathcal{K}(\phi \circ f \circ \phi^{-1})$ and $\phi([\mathcal{K}(f)]) = [\phi(\mathcal{K})] = [\mathcal{K}(\phi \circ f \circ \phi^{-1})]$. \diamond

We can restate the following result from [PM3] in terms of germs of hedgehogs:

Theorem 2.8 ([PM3]). *If two nonlinearisable germs f, g commute then any sufficiently small hedgehog of f is also a hedgehog of g , so they define the same germ of hedgehog $[\mathcal{K}(f)] = [\mathcal{K}(g)]$.*

3. Results.

The key result from which the others follow easily is the following:

Theorem 3.1. *Let K be a hedgehog for a nonlinearizable germ f . If a germ $T(z) = z + O(z^2)$ tangent to the identity is univalent on a neighbourhood of K and K is either forward or backward invariant for T , then T is equal to the identity.*

For notational convenience, given quantities a, b which are either positive sequences or functions of z near 0, we write $a \preceq b$ if $a \leq Cb$ for some constant $C > 0$ for all n sufficiently large (or all z sufficiently small as the case may be). We write $a \sim b$ if $a \preceq b$ and $b \preceq a$.

Proof: Suppose $T \neq \text{id}$, then $T(z) = z + c_d z^{d+1} + O(z^{d+2})$ (for some $d \geq 1, c_d \neq 0$) is a nondegenerate parabolic germ. Let $U_1 \subset U_2, U_1 \neq U_2$ be admissible domains for f and T such $K \cap \partial U_1 \neq \emptyset, K = K(U_1)$. Let $F \subset \overline{U_2}$ be a Siegel compact of T such that $F \cap \partial U_2 \neq \emptyset$. As Perez-Marco shows in [PM1], F is an invariant Fatou flower of T , meaning that $\text{int}(F) = P_1 \cup \dots \cup P_d$ where P_1, \dots, P_d (the 'petals' of the 'flower') are pairwise disjoint Jordan domains invariant under T whose boundaries intersect only at the origin, and $T|_{P_i}$ is a parabolic automorphism of P_i having 0 as the unique fixed point on the boundary. Pick a point $z_0 \in \text{int}(F) - \overline{U_1}$ and a small ball B_0 around z_0 contained in $\text{int}(F) - \overline{U_1}$. Since $B_0 \subset P_i$ for some i , $T^n(B_0) \rightarrow \{0\}$ as $n \rightarrow \pm\infty$.

We may assume wlog (considering T^{-1} if necessary) that $T(K) \subset K$. For $n \geq 0$ let $z_n = T^{-n}(z_0), B_n = T^{-n}(B_0)$. Then points in B_n escape from $\overline{U_1}$ under T^n , while by hypothesis points of the hedgehog K remain in $K \subset \overline{U_1}$, so we will arrive at the desired contradiction if we can show that $B_n \cap K \neq \emptyset$ for some n . We need the following estimate on the asymptotic size of the B_n 's:

Lemma 3.2. $d(z_n, \partial B_n) \succeq |z_n|^{d+1}$

Proof: Taking a covering of a neighbourhood of the origin by attracting and repelling Fatou petals, we see that the domains B_n converge to 0 through a repelling petal P ; there is a Fatou coordinate $w = \chi(z)$ defined on P which maps P to the right half-plane $\{\operatorname{Re} w > 0\}$ and conjugates T^{-1} to the translation $w \mapsto w + 1$. Moreover, χ has an asymptotic expansion of the form $\chi(z) = c \log(z) + \sum_{n \geq -d} c_n z^n$ with $c_{-d} \neq 0$ (see for example [Ec]), so $|\chi'(z)| \sim |z|^{-(d+1)}$ and $|(\chi^{-1})'(w)| \sim |w|^{-1/d-1}$. Fix an n_0 such that $B_{n_0} \subset P$. Note that for $n > n_0$, all the domains $\chi(B_n)$ are translates of $\chi(B_{n_0})$ and have the same constant diameter, so for large n , for all $w \in \chi(B_n)$, $|w| \sim |\chi(z_n)|$, and hence $|(\chi^{-1})'(w)| \sim |(\chi^{-1})'(\chi(z_n))| = |\chi'(z_n)|^{-1} \sim |z_n|^{d+1}$. It follows that for all $z \in B_n$, for n large, putting $w = \chi(z)$, $|\chi'(z)|^{-1} = |(\chi^{-1})'(w)| \sim |z_n|^{d+1}$. So taking $z'_n \in \partial B_n$ such that $|z_n - z'_n| = d(z_n, \partial B_n)$, for $n > n_0$ we have, for some $z''_n \in B_n$,

$$\begin{aligned} d(z_n, \partial B_n) &= |z_n - z'_n| \geq |\chi'(z''_n)|^{-1} |\chi(z_n) - \chi(z'_n)| \\ &\sim |z_n|^{d+1} d(\chi(z_n), \partial \chi(B_n)) \\ &= |z_n|^{d+1} d(\chi(z_{n_0}), \partial \chi(B_{n_0})) \end{aligned}$$

and the Lemma follows. \diamond

The points z_n converge slowly to 0, in the sense that $|z_{n+1}|/|z_n| = |T^{-1}(z_n)|/|z_n| = 1 + O(z_n^d) \rightarrow 1$ as $n \rightarrow \infty$. To prove the Theorem it suffices to prove the following Proposition:

Proposition 3.3 *Let K be a hedgehog of a nonlinearizable germ f . Let (z_n) be a sequence converging to 0 such that for n large enough $|z_{n+1}| \geq \epsilon |z_n|$ for some $\epsilon > 0$, and (B_n) a sequence of domains such that $z_n \in B_n$ and $d(z, \partial B_n) \succeq |z_n|^{d+1}$ for some $d \geq 1$. Then for some subsequence (B_{n_k}) , $K \cap B_{n_k} \neq \emptyset$ for all large k .*

Proof: Note that for any change of variables $\phi(z) = z + O(z^2)$, the sequences $(\phi(z_n))$ and $(\phi(B_n))$ satisfy the hypotheses of the Proposition, thus we may assume wlog, that f is of the form $f(z) = e^{2\pi i \alpha} z + O(z^N)$ for some large N which we will choose appropriately in the course of the proof (any irrationally indifferent germ can always be analytically conjugated to germs tangent to the rotation R_α upto arbitrarily high orders). We need the following two estimates on how long we can iterate such a germ close to the origin, and how close the orbits stay to that of the rotation:

Lemma 3.4 *Given $f(z) = e^{2\pi i \alpha} z + O(z^N)$ for some $N \geq 2$, for all z small enough, at least $M(z)$ iterates of f are defined on z where $M(z) = C|z|^{-N+1}$ for some $C > 0$, and moreover*

$$|f^k(z)| \leq 2|z|, \quad k = 0, \dots, M(z).$$

Proof: There are constants $C_1 > 0$ and $\epsilon_0 > 0$ such that for $|z| \leq \epsilon_0$ we have

$$|f(z) - e^{2\pi i \alpha} z| \leq C_1 |z|^N.$$

So for $|z| \leq \epsilon_0$,

$$|f(z)| \leq |z| + C_1 |z|^N = \phi(|z|) \quad - (1)$$

where $\phi(t) = t + C_1 t^N$. To estimate $|f^k(z)|$ for small z , we estimate $\phi^k(t)$ for $t = |z|$ close to 0. It is convenient to conjugate the mapping $t \mapsto \phi(t)$ to a mapping $s \mapsto \tilde{\phi}(s)$, in terms of the variables $s = 1/t^{N-1}$ and $\tilde{\phi}(s) = 1/\phi(t)^{N-1}$ close to $+\infty$. A calculation gives

$$\tilde{\phi}(s) = s - (N-1)C_1 + O(1/s) \geq s - C_2$$

for $s \geq s_0$ sufficiently large, for some constants s_0, C_2 . It follows that for $s \geq 2s_0$ and $k \leq s/(2C_2)$, we have

$$\tilde{\phi}^k(s) \geq s - kC_2$$

In terms of the variables $\phi(t), t$, this means that for $t \leq t_0$ sufficiently small and $k \leq 1/(2C_2 t^{N-1})$,

$$\phi^k(t) \leq t(1 - kC_2 t^{N-1})^{\frac{-1}{N-1}}$$

For $k \leq 1/(2C_2 t^{N-1})$, it is easy to see from the above that $\phi^k(t) \leq 2t$. So for $|z| \leq \min(t_0, \epsilon_0/2)$, we have $\phi^k(|z|) \leq \epsilon_0$ for $k \leq C|z|^{-N+1}$ where $C = 1/(2C_2)$. Since ϕ is monotone increasing, it follows from (1) by induction that

$$|f^k(z)| \leq \phi^k(|z|) \leq 2|z| \leq \epsilon_0, k = 0, \dots, C|z|^{-N+1}$$

so that at least $M(z) = C|z|^{-N+1}$ iterates of f on z are defined. \diamond

Lemma 3.5. *For $|z| \leq \epsilon$, we have*

$$|f^k(z) - R_\alpha^k(z)| \leq kC_2|z|^N, \quad k = 0, \dots, M(z)$$

for some $C_2 > 0$.

Proof: Since $|f^k(z)| \leq 2|z| \leq \epsilon_0$ for $|z| \leq \epsilon, k = 0, \dots, M(z)$, letting $z_k = f^k(z)$, we know

$$|f(z_k) - R_\alpha(z_k)| \leq C_1|z_k|^N, \quad k = 0, \dots, M(z)$$

so

$$\begin{aligned} |f^k(z) - R_\alpha^k(z)| &= \left| (f(z_{k-1}) - R_\alpha(z_{k-1})) + \sum_{j=1}^{k-1} (R_\alpha^j(z_{k-j}) - R_\alpha^{j+1}(z_{k-j-1})) \right| \\ &\leq |(f(z_{k-1}) - R_\alpha(z_{k-1}))| + \sum_{j=1}^{k-1} |(R_\alpha^j(z_{k-j}) - R_\alpha^{j+1}(z_{k-j-1}))| \\ &= |(f(z_{k-1}) - R_\alpha(z_{k-1}))| + \sum_{j=1}^{k-1} |(R_\alpha^j(g(z_{k-j-1})) - R_\alpha^j(R_\alpha(z_{k-j-1})))| \\ &= \sum_{j=0}^{k-1} |f(z_j) - R_\alpha(z_j)| \\ &\leq \sum_{j=0}^{k-1} C_1|z_j|^N \\ &\leq kC_2|z|^N \end{aligned}$$

since $|z_j| \leq 2|z|$, $j \leq k-1 \leq M(z)$. \diamond

Proof of Proposition 3.3: For all n sufficiently large, the circle $\{|z| = |z_n|\}$ intersects the hedgehog K since it is a non-trivial connected set containing the origin; let w_n be a point of K on this circle. Let $(p_k/q_k)_{k \geq 0}$ be the continued fraction convergents of α . For all k , it follows from the continued fraction algorithm that any point on the circle $\{|z| = |z_n|\}$ is at distance at most $2q_k^{-1}2\pi|z_n|$ from the first q_k points $R^m(w_n)$, $m = 0, \dots, q_k$ of the orbit of w_n under the rotation R_α . If for each k we take z_{n_k} to be the first element of the sequence (z_n) such that $|z_{n_k}| < q_k^{-1/(d+1)}$, so $q_k^{-1/(d+1)} \leq |z_{n_k-1}|$ then by hypothesis $|z_{n_k}| \geq \epsilon|z_{n_k-1}| \geq \epsilon q_k^{-1/(d+1)}$, thus $|z_{n_k}| \sim q_k^{-1/(d+1)}$, and $2q_k^{-1}2\pi|z_{n_k}| \sim |z_{n_k}|^{d+1}|z_{n_k}| = |z_{n_k}|^{d+2}$. So for all k there is some $0 \leq m_k \leq q_k$ such $|z_{n_k} - R^{m_k}(w_{n_k})| \preceq |z_{n_k}|^{d+2}$. By Lemma 3.5, the orbit of w_{n_k} under f stays close to the orbit under R_α for at least $M(w_{n_k})$ iterates, and $M(w_{n_k}) \sim |w_{n_k}|^{-(N-1)} = |z_{n_k}|^{-(N-1)} \sim q_k^{(N-1)/(d+1)}$, so assuming $N \geq d+2$ we have $M(w_{n_k}) \geq q_k$ for all large k , and $|f^{m_k}(w_k) - R^{m_k}(w_k)| \leq q_k C_2 |z_{n_k}|^N \sim |z_{n_k}|^{-(d+1)} |z_{n_k}|^N = |z_{n_k}|^{N-(d+1)}$. Thus by taking $N \geq 2d+3$ we have $|f^{m_k}(w_k) - R^{m_k}(w_k)| \preceq |z_{n_k}|^{d+2}$, and

$$\begin{aligned} |z_{n_k} - f^{m_k}(w_k)| &\leq |z_{n_k} - R^{m_k}(w_{n_k})| + |f^{m_k}(w_k) - R^{m_k}(w_k)| \\ &\preceq |z_{n_k}|^{d+2} \end{aligned}$$

Since $d(z_{n_k}, \partial B_{n_k}) \succeq |z_{n_k}|^{d+1}$, it follows that $f^{m_k}(w_k) \in B_{n_k}$ for all large k and $K \cap B_{n_k} \neq \emptyset$. \diamond

Proof of Theorem 3.1. By Proposition 3.3 we can pick a point w in $K \cap B_n$ for some $n > 0$; then $T^n(w) \in B_0 \subset \overline{U_1^c}$ whereas by hypothesis $T^n(w) \in K \subset \overline{U_1}$, a contradiction. \diamond

Theorem 3.6. *A nonlinearizable germ f cannot commute with a nondegenerate parabolic germ g .*

Proof: While this is easy to see using formal power series arguments, we can give a dynamical proof using the previous result. Indeed if f and g commute, then for a small hedgehog K of f on a neighbourhood of which g, g^{-1} are univalent, $f(g(K)) = g(f(K)) = g(K)$, so $g(K)$ is also a hedgehog for f . Since the hedgehogs of f are linearly ordered with respect to inclusion, K is either forward or backward invariant under g , contradicting Theorem 3.1, since g^q is tangent to the identity for some q but $g^q \neq id$. \diamond

Theorem 3.7. *Two nonlinearisable germs f_1 and f_2 are holomorphically conjugate by a germ ϕ if and only if ϕ maps some hedgehog K_1 of f_1 to a hedgehog K_2 of f_2 and the rotation numbers of f_1 and f_2 are equal. Thus for a nonlinearisable germ f its rotation number and germ of hedgehog are a complete set of holomorphic conjugacy invariants.*

Proof: If $f_2 = \phi \circ f_1 \circ \phi^{-1}$, then for a hedgehog K_1 of f_1 ,

$$f_2(\phi(K_1)) = \phi \circ f_1 \circ \phi^{-1}(\phi(K_1)) = \phi(f_1(K_1)) = \phi(K_1)$$

which implies that $\phi(K_1)$ is a hedgehog of f_2 . This proves the "only if" part.

For the "if" part, given $\phi(K_1) = K_2$, let \tilde{f}_2 be the conjugate $\tilde{f}_2 := \phi \circ f_1 \circ \phi^{-1}$. Then $\tilde{f}_2(K_2) = K_2$, so the germ $g = \tilde{f}_2 \circ f_2^{-1}$ preserves K_2 . Since the rotation numbers of f_1 and f_2 are equal, g is tangent to the identity and hence by Theorem 3.1 is equal to the identity. So $\tilde{f}_2 = f_2$, i.e. ϕ conjugates f_1 to f_2 . \diamond

We have the converse of Theorem 2.8:

Theorem 3.8. *If two nonlinearizable germs f and g have a common hedgehog K then they commute. In particular if f and g have the same germ of hedgehog (so all sufficiently small hedgehogs of one are hedgehogs of the other) then they commute.*

Proof: The hedgehog K is invariant under the commutator $f \circ g \circ f^{-1} \circ g^{-1}$ which is tangent to the identity and therefore by Theorem 3.1 must be the identity. \diamond

We now consider abelian subgroups H of $\text{Diff}(\mathbf{C}, 0)$ such that H contains at least one irrationally indifferent germ. If any irrationally indifferent germ in H is linearizable then all germs in H are linearizable and we call H linearizable, otherwise all irrationally indifferent germs in H are nonlinearizable (the rationally indifferent germs in H are of finite order and linearizable) and we call H nonlinearizable. Given a nonlinearizable abelian subgroup H , by Theorem 2.8 all irrationally indifferent germs in H determine the *same* germ of hedgehog, which we denote by $[\mathcal{K}(H)]$. Conversely, to every germ of hedgehog $[\mathcal{K}]$, we can associate the subgroup of germs which leave it invariant, $\text{Aut}([\mathcal{K}]) := \{f \in \text{Diff}(\mathbf{C}, 0) : f([\mathcal{K}]) = [\mathcal{K}]\}$.

Theorem 3.9 *For any germ of hedgehog $[\mathcal{K}]$, the homomorphism*

$$\begin{aligned} \lambda : \text{Aut}([\mathcal{K}]) &\rightarrow \mathbf{C}^* \\ g &\mapsto g'(0) \end{aligned}$$

is an injective homomorphism into $S^1 \subset \mathbf{C}^$. Thus $\text{Aut}([\mathcal{K}])$ is abelian and germs in $\text{Aut}([\mathcal{K}])$ are uniquely determined by their rotation numbers.*

Proof: For any $g \in \text{Aut}([\mathcal{K}])$, for all sufficiently small $K \in \mathcal{K}$, $g(K) \in \mathcal{K}$, and since hedgehogs of a nonlinearizable germ are linearly ordered, K is either forward or backward invariant under g . Any g in the kernel of λ is tangent to the identity and by Theorem 3.1 equals the identity. Thus λ is injective and $\text{Aut}([\mathcal{K}])$ is abelian. If $|g'(0)| \neq 1$ for some $g \in \text{Aut}([\mathcal{K}])$ then g is linearizable, and since $\text{Aut}([\mathcal{K}])$ is abelian, the linearization of g linearizes all germs in $\text{Aut}([\mathcal{K}])$, contradicting the existence of at least one nonlinearizable germ in $\text{Aut}([\mathcal{K}])$. Thus $|g'(0)| = 1$ for all $g \in \text{Aut}([\mathcal{K}])$. \diamond

Finally we have:

Theorem 3.10. *There is a bijective correspondence between nonlinearizable maximal abelian subgroups of $\text{Diff}(\mathbf{C}, 0)$ and germs of hedgehogs:*

$$\begin{aligned} H &\rightarrow [\mathcal{K}(H)] \\ \text{Aut}([\mathcal{K}]) &\leftarrow [\mathcal{K}] \end{aligned}$$

The action of $\text{Diff}(\mathbf{C}, 0)$ on nonlinearisable maximal abelian subgroups by conjugation $(h, H) \mapsto h H h^{-1}$ corresponds to the action of $\text{Diff}(\mathbf{C}, 0)$ on germs of hedgehogs $(h, [\mathcal{K}]) \mapsto$

$h([\mathcal{K}]) : H_1 = h H_2 h^{-1}$, if and only if $h([\mathcal{K}(H_1)]) = [\mathcal{K}(H_2)]$. Thus the germ of hedgehog $[\mathcal{K}(H)]$ of a nonlinearizable maximal abelian subgroup H is a complete conjugacy invariant of H .

Proof: We first check that every subgroup $\text{Aut}([\mathcal{K}])$ is maximal abelian. Let $f \in \text{Aut}([\mathcal{K}])$ be a nonlinearizable germ such that $[\mathcal{K}(f)] = [\mathcal{K}]$. Any germ g commuting with $\text{Aut}([\mathcal{K}])$ commutes with f and hence maps sufficiently small hedgehogs K of f to hedgehogs of f since $f(g(K)) = g(f(K)) = g(K)$, thus $g([\mathcal{K}]) = [\mathcal{K}]$ and $g \in \text{Aut}([\mathcal{K}])$.

It is straightforward to check that the two maps are mutual inverses. The second assertion of the Theorem follows easily from Theorem 3.7. \diamond

Remark. It is well known that the unique maximal abelian subgroup containing an irrationally indifferent germ f is its centralizer $\text{Cent}(f)$, thus $\text{Aut}([\mathcal{K}]) = \text{Cent}(f)$ for any nonlinearizable f whose germ of hedgehog is $[\mathcal{K}]$.

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